Equilibrium Selection with Nonlinear Utility Function (Financial Modeling and Analysis)

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Equilibrium Selection with Nonlinear Utility Function [1)]

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Abstract

This paper examines whether or not that each player's utility function is non-linear in a general game. First, we review evolutionary game theory. Next, we examine equilibrium selection and prove the approachable under the risk with nonlinear utility function. Furthermore, we prove that the strategic distribution is a log-normal distribution in a random environment.

1 Introduction

We have asserted that a utility function is linear in evolutionary game theory. However, we can understand that it is not necessary to assume that a utility function is linear.

This research expands the following account is taken into consideration. On the notion, a utility function is nonlinear and we examine the equilibrium selection, the strategy of which is Nash equilibrium, with this nonlinear utility function. This research examines and explains famous paradoxes[3)] in the expected utility theory with nonlinear utility function. This nonlinear utility function can demonstrate that each player has a "risk attitude", "emotion" and the emergence of "altruistic behavior".[4)] Especially, this research discusses "risk" in this context.[5)]

This paper is organized as follows. In Section 2, we review traditional evolutionary game theory. In Section 3, we examine equilibrium selection with nonlinear utility function and we prove the approachable under risk. In Section 4, we extend the context of Section 3 and we examine the distribution of the strategy in a random environment. In Section 5, we present the conclusions and discuss future research.

2 Preliminary: Evolutionary Game Theory

In traditionary evolutionary game theory, each player chooses a strategy randomly. Or, alternatively, a large number of players is assumed to search at random for a game, and when they

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[3)]An inconsistency of actual observed choices with the predictions of expected utility theory.

[4)]Sethi and Somanathan [14] examines the common pool resource game and derives the conditions that tragedy of commons does not occur with Levine [9]'s altruistic utility function. This paper shows that the equilibrium is changed when a utility function is changed.

[5)]There is some related literature : Karni and Schmeidler [5] derive that the maximization of probability of survival is consistent with maximization of the expected utility function. On the other hand, Robson [11, 12, 13] examines which strategy is Nash equilibrium in a random environment which corresponds to a risk. In this case, this result does not always coincide with Karni and Schmeidler [5].

meet, the terms of the game are started.

First, we formulate the game. A **strategic game** is $G = \big(N, \{Q_i\}_{i\in N}, \{g_i\}_{i\in N}\big)$, where $N = \{1, 2, \cdots, n\}$ is the set of **players**, Q_i is the set of **strategies/actions** available to player i. All the players' strategies are expressed by $\vec{q} = q_1, \cdots, q_n$. The strategy q_i is called a **pure strategy**. g_i is a measurable function from the product set $\vec{Q} = Q_1 \times \cdots \times Q_n$ to a real number and this is represented by a player i's **utility function**.

We define the equilibrium concept in evolutionary game theory.

Definition 1 *$q_i \in Q_i$ is an* ***evolutionarily stable strategy*** *(ESS) if for every strategy $q_j \neq q_i$, there exists some $\bar{\epsilon}_q \in (0, 1)$ such that the following inequality holds for all $\varepsilon \in (0, \bar{\epsilon}_q)$*

$$g\big[q_i, \varepsilon q_j + (1-\varepsilon)q_i\big] > g\big[q_j, \varepsilon q_j + (1-\varepsilon)q_i\big]. \tag{2.1}$$

This definition is characterized by the following proposition.

Proposition 1 *(Bishop and Cannings [2]) $q_i \in Q_i$ is an evolutionarily stable strategy if and only if it meets these first-order and second-order best-replies:*

$$g(q_j, q_i) \leq g(q_i, q_i), \quad \forall q_j, \tag{2.2}$$

$$g(q_j, q_i) = g(q_i, q_i) \quad \Rightarrow \quad g(q_j, q_j) < g(q_i, q_j), \quad \forall q_j \neq q_i. \tag{2.3}$$

Proof. See Weibull [15]. □

We can understand that (2.2) is a Nash equilibrium condition, (2.3) is an asymptotically stable condition. Thus, ESS expresses the stable state in the system.

Next, we formulate the dynamic process. Let $x_i(t) = \dfrac{p_i(t)}{P(t)}$ be the probability of choosing the strategy $i \in N$, or the population share of choosing the strategy i, where $P(t)$ is the whole population.[6)] Let $p_i(t)$ be the population of choosing the strategy i and g_i be the growth rate in the population $p_i(t)$.

We examine the variation of the $x_i(t)$: $x_i(t+\Delta t) = \dfrac{p_i(t+\Delta t)}{P(t+\Delta t)}$. We can obtain as follows.

$$x_i(t+\Delta t) = \frac{x_i(t+\Delta t)P(t+\Delta t)}{P(t+\Delta t)} = \frac{(1+g_i)x_i(t)P(t)}{P(t+\Delta t)} = \Big[\frac{1+g_i}{1+\bar{g}}\Big]x_i(t), \ \ \bar{g} = \sum_{i=1}^{N} x_i g_i.$$

If we examine the difference at intervals Δt from the above equation, we can obtain as follows.

$$x_i(t+\Delta t) - x_i(t) = x_i(t)\Big[\frac{1+g_i}{1+\bar{g}} - 1\Big] = x_i\Big[\frac{1+g_i-1-\bar{g}}{1+\bar{g}}\Big] = x_i\Big[\frac{g_i-\bar{g}}{1+\bar{g}}\Big].$$

As $\Delta t \to 0$, we can obtain as follows.

6) The whole population is finite. But law of the large numbers is realized in this population size. If the whole population is infinite, the definitions of game theory are more difficult.

$$\dot{x}_i = x_i(g_i - \bar{g}). \tag{2.4}$$

This is called a **Replicator equation**. A replicator equation means that if the player's payoff from the outcome i is greater than the expected utility $x \cdot Ax$, then the probability of the action i is higher than before. There is an **externality** : if another player's probability of choosing the strategy is greater, one's own probability of choosing the strategy is greater.

If the utility function is linear : $g_i(z) = z$ (Payoff Matrix 1), replicator equation in symmetric game with two strategies is as follows.

I \\ II	Strategy 1	Strategy 2
Strategy 1	a, a	$0, 0$
Strategy 2	$0, 0$	b, b

Payoff matrix 1

$$\dot{x} = x(1-x)\{ax - b(1-x)\} \tag{2.5}$$

In this payoff matrix 1, we can classify Nash equilibrium which depends on the signs of the payoff: a, b.

Remark 1 ***i)*** *If $a > 0, b < 0$, then we have a game of the* ***Non-Dilemma*** *variety, and the game has exactly one Nash equilibrium. This equilibrium is strict and symmetric. Hence such a game poses exactly one ESS: (strategy 1, strategy 1).*

ii) *If $a < 0, b > 0$, then we have a game of the* ***Prisoner's Dilemma*** *variety, and the game has exactly one Nash equilibrium. This equilibrium is strict and symmetric. Hence such game poses exactly one ESS: (strategy 2, strategy 2) like a i). This equilibrium is Pareto inferior.*

iii) *If $a > 0, b > 0$, then we have a* ***Coordination Game****, and there are three Nash equilibria : two pure strategies, mixed strategy. Each of the two pure equilibria are evolutionary stable.*

iv) *If $a < 0, b < 0$, then we have a* ***Hawk-Dove Game****. Such a game has two strict asymmetric Nash equilibria (pure strategies) and one symmetric Nash equilibrium (mixed strategy). Mixed strategy is evolutionary stable.*

3 Nonlinear Utility Function

In Section 2, we review the elements in evolutionary game theory. As we know (2.4), utility function g_i is a general function and this function is not defined as linear or nonlinear.

So, we assume that utility function is linear, $g_i(z) = z$ in traditional evolutionary game theory. In this research, we discuss the impact of risk with each utility function's first and second order Taylor expansion.

We assume that $g(z)$ is nth continuously differentiable function. The utility function $g(w+z)$ is as follows.

$$g(w+z) = g(w) + g'(w)z + \frac{1}{2}g''(w)z^2 + O(z^3). \tag{3.1}$$

where $z \in W$ (W is the commodity bundle) is a payoff in this game, $w \in W$ is the value of own assets, So, w expresses own wealth. We can understand one's own utility though this game as follows.

$$U(z) \equiv g(w+z) - g(w) = g'(w)z + \frac{1}{2}g''(w)z^2 + O(z^3) \tag{3.2}$$

We examine the following paradox, which is not explained by linear utility function with the above utility function.

Example 1 *(St. Petersburg paradox)*

In a game of chance, you pay a fixed fee to enter, and then a fair coin will be tossed repeatedly until tails first appears, ending the game. The pot starts at 2 dollars and is doubled every time heads appears. You win whatever is in the pot after the game ends. Thus you win 2 dollars if tails appears on the first toss, 4 dollars if heads appears on the first toss and tails on the second, etc. In short, you win $2k-1$ dollars if the coin is tossed k times until the first tails appears.

What would be a fair price to pay for entering the game?[7] To answer this we need to consider what would be the average payoff.

$$\frac{1}{2}\{g'(w)2 + \frac{1}{2}g''(w)2^2\} + \left(\frac{1}{2}\right)^2\{g'(w)2^2 + \frac{1}{2}g''(w)2^4\} + \cdots + \left(\frac{1}{2}\right)^n\{g'(w)2^n + \frac{1}{2}g''(w)2^{2n}\}$$
$$= ng'(w) + (2^n-1)g''(w).$$

If n is infinite, the expected value depends on the sign and value of $g'(w), g''(w)$ and this has a convergence. However, if the utility function is linear, the expected value is infinite.

3.1 Equilirbium Selection

In this section, we consider the symmetric two person game with two strategies. We assume that this game's payoff is the following.

I \ II	Strategy 1	Strategy 2
Strategy 1	f_1, f_1	$0, 0$
Strategy 2	$0, 0$	f_2, f_2

Payoff Matrix 2

However, if we use the above utility function (3.2), the payoff is changed as follows. For example, it is $f_1 = g'(w)f_1$ with the first order Taylor expansion and it is $f_1 = g'(w)f_1 + \frac{f_1^2}{2}g''(w)$ with the second order Taylor expansion.

[7]If we consider this question with expected utility theory, we need to consider what would be the average payoff. With probability 1/2, you win 2 dollars; with probability 1/4 you win 4 dollars, etc. We can calculate that this expected value is infinite.

3.1.1 Utility function: a first order Taylor expansion

We examine the equilibrium selection with the first order Taylor expansion ($g(w+z)-g(w) = g'(w)z$) like a Remark 1.

Proposition 2 ***i)*** *If $f_1 > 0, f_2 < 0$: $g'(w) > 0, a > 0, b < 0$ or $g'(w) < 0, a < 0, b > 0$, then we have a Non-Dilemma Game.*

ii) *If $f_1 < 0, f_2 > 0$: $g'(w) > 0, a < 0, b > 0$ or $g'(w) < 0, a > 0, b < 0$, then we have a Prisoner's Dilemma Game.*

iii) *If $f_1 > 0, f_2 > 0$: $g'(w) > 0, a > 0, b > 0$ or $g'(w) < 0, a < 0, b < 0$, then we have a Coordination Game.*

iv) *If $f_1 < 0, f_2 < 0$: $g'(w) > 0, a < 0, b < 0$ or $g'(w) < 0, a > 0, b > 0$, then we have a Hawk-Dove Game.*

Proof. We can prove this proposition easily with the same method as in Remark 1. □

Thus we can understand the following. If $g'(w)$ is positive, this result is the same as the linear case. But if $g'(w)$ is negative, this result is opposite to the linear case.

3.1.2 Utility function: a second order Taylor expansion

We examine the equilibrium selection with the second order Taylor expansion $\big(g(w+z)-g(w) = g'(w)z + \frac{z^2}{2}g''(w)\big)$ like Remark 1 and Proposition 2. Here, it is convenient to redefine the payoff $(g(w+z)-g(w))$ with the following definition.

Definition 2 *(Arrow [1], Pratt [10]): Given a (twice-differentiable) Bernoulli utility function $u(\cdot)$ for money, the* ***Arrow-Pratt coefficient of absolute risk aversion*** *at x is defined as*

$$r_A(x) = -\frac{u''(x)}{u'(x)}. \tag{3.3}$$

We use the above definition. If $z\Big(1-\frac{z}{2}r_A(w)\Big) > 0$, then $g(w+z)-g(w) > 0$. We can understand that a player obtains a positive payoff. We examine the Allais paradox with this definition.

Example 2 *(Allais paradox)*

There are two lotteries (lottery 1 and 2) and two choices/strategies for each lottery. We consider which lotteries each player prefers.

Lottery 1: we can receive the money: 1 million yen with probability 1.

Lottery 1': we can receive the money: 0 yen with probability 0.01, 5 million yen with probability 0.10 and 1 million yen with probability 0.89.

Allais asserted that most people would choose Lottery 1. Next we consider the following lottery.

Lottery 2: we can receive the money: 1 million yen with probability 0.11, 0 yen with probability 0.89.

Lottery 2': we can receive the money: 5 million yen with probability 0.10, 0 yen with probability 0.90.

Allais asserted that most people would choose Lottery 2'.

The first choice means that one prefers the certainty of receiving 1 million yen over a lottery offering a 1/10 probability of getting five times more but bringing with it a tiny risk of getting nothing. The second choice means that, all things considered, a 1/10 probability of getting 5 million yen is preferred to getting only 1 million yen with slightly better odds of 11/100.

These choices are not consistent with linear utility function. However, we can explain these choices with nonlinear utility function. If $r_A > \frac{0.39}{1.195}$, one prefers the lower expected utility. If $r_A < \frac{0.39}{1.195}$, one prefers the higher expected utility. We can understand that with the higher value of r_A, one prefers the more risky choice.

Next, we examine the equilibrium selection with the nonlinear utility function. But, it is difficult to check the sign of utility owing to four variables $(a, b, g'(w), g''(w))$. So, we examine the limited case.

Example 3 *We examine the traditional economics situation :* $g'(w) > 0, g''(w) < 0$.[8)]

(i) If $z > 0$ and $zr_A(w) < 2$, then $g(w+z) - g(w) > 0$. If $z > 0$ and $zr_A(w) < 2$, then $g(w+z) - g(w) < 0$.

(ii) If $z < 0$, then $g(w+z) - g(w) < 0$.

We can understand the following propositon with these properties.

Proposition 3 ***i)*** *If* $f_1 > 0, f_2 < 0$: $g'(w) > 0, g''(w) < 0$ *and* $ar_A(w) < 2$, $b < 0$, *then we have a Non-Dilemma Game.*

ii) $f_1 < 0, f_2 > 0$: $g'(w) > 0, g''(w) < 0$ *and* $a < 0$, $br_A(w) < 2$, *then we have a Prisoner's Dilemma Game.*

iii) $f_1 > 0, f_2 > 0$: $g'(w) > 0, g''(w) < 0$ *and* $ar_A(w) < 2$, $br_A(w) < 2$, *then we have a Coordination Game.*

iv) $f_1 < 0, f_2 < 0$: $g'(w) > 0, g''(w) < 0$ *and* $ar_A(w) < 2$, $br_A(w) < 2$, *or* $ar_A(w) > 2$ *and* $br_A(w) > 2$, $ar_A(w) > 2$ *and* $b < 0$ *or* $a < 0$ *and* $br_A(w) > 2$, *then we have a Hawk-Dove Game.*

Proof. We can prove this proposition easily with the same method as in Remark 1. □

If the risk is high under the traditional utility function, then each player supposes that the game is the Hawk-Dove type. A mixed strategy is adopted by each player.

8) We can consider that z is negative. This situation is similar to Kaheman and Tversky [4]'s.

3.2 Replicator Equation with Non-linear Utility

In this section, we examine the dynamic impact of the nonlinear utility function. We introduce a replicator equation into the nonlinear utility function.

Proposition 4 *A mixed strategy equilibrium of the game is approachable under risk r_A.*[9]

Proof. See appendix.

Thus, we can understand that the mixed strategy becomes adopted by each player as in Proposition 3.

4 Extension: Random Environment

In this section, we examine the impacts of environmental variation on the game. Here, environmental variation corresponds to the payoff variation. Kikkawa [6] proves that a strategy's distribution is a log-normal distribution and the game with the varying payoff is approachable under the variance σ in a random environment. We extend Kikkawa [6] as for each player's utility, and examine the impact of the risk attitude in a random environment.

For easy discussion, we assume that the variation payoff is a normal distribution. We can obtain the following proposition in this game.

Proposition 5 *A strategy distribution x is a log-normal distribution in this game.*

Proof. See appendix.

This result is similar to Kikkawa [6]. But the average and dispersion in this game are depends on the sign and magnitude of $g'(w), g''(w)$.

5 Concluding Remarks

We have examined the equilibrium selection, proved the approachable under risk, and the strategy distribution is log-normal distribution in a random environment with nonlinear utility function.

This research examines several things under the complete information. There are future works about incomplete information for theoretically. We can apply this game to the financial market. In the financial market, a stock motion is a Brownian motion and we can consider that the payoff is changing randomly. We can construct a model with each player's micro-foundation in mathematical finance. (Kikkawa [7, 8])

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[9] Harsanyi [3] used the phrase "approachable". This means that when the random variations in payoffs are small, almost any mixed equilibrium of the game is close to a pure equilibrium.

Appendix

Proof of Proposition 4

If we introduce replicator equation (2.5) into nonlinear utility function (3.2) and transform, we can obtain as follows.

$$\dot{x} = x(1-x)\Big\{(a+b)g'(w) + \frac{1}{2}(a^2+b^2)g''(w) - bg'(w) + \frac{1}{2}b^2g''(w)\Big\}. \tag{5.1}$$

If we derive the equilibrium from this equation, we can obtain three equilibrium points : $(x^*, 1 - x^*) = (0,1), (1,0), \Big(\frac{b - \frac{r_A}{2}b^2}{a+b-\frac{r_A}{2}(a^2+b^2)}, 1 - \frac{b - \frac{r_A}{2}b^2}{a+b-\frac{r_A}{2}(a^2+b^2)}\Big)$.

As we know, there is only interior equilibrium to impact the equilibrium for a risk. We can understand that this interior equilibrium is approachable under absolute risk aversion r_A.

This research is similar to Harsanyi [3]. If each player receives a payoff at a strategy is subject to each player's risk, each player knows the realization of r_A but not the realizations of the other player's risk. So each player chooses a mixed strategy. □

Proof of Proposition 5

If we introduce replicator equation (2.5) into nonlinear utility function (3.2) and transform, we can obtain as follows.

$$\frac{\dot{x}_i}{x_i} = zg'(w) + \frac{z^2}{2}g''(w) - \bar{g}, \tag{5.2}$$

where the average payoff $\bar{g} = E[g_i] = E(z)g'(w) + E\Big[\frac{z^2}{2}\Big]g''(w) = \frac{\sigma_z^2}{2}g''(w)$.

Let time step t divided $n\tau$, τ is the short time scale, n is integer. Let the integral of each short interval be $\xi_k = \int_{(k-1)\tau}^{k\tau} \xi(t)dt$. $\xi_k (k = 1, 2, \cdots, n)$ are n-tuples random variable with the mean value 0.

We can transform (5.2) as follows.

$$\log \frac{x(t)}{x(0)} = \big(g'(w) + \sigma_z g''(w)\big)\sum_{k=1}^{n} \xi_k + \frac{g''(w)}{2}\sum_{k=1}^{n}(\xi_k - \sigma_z)^2 \tag{5.3}$$

If $n \to \infty$ in the above equation, central limit theorem is realized.

Theorem A.1. *(central limit theorem) Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with finite mean m and finite a non-zero variance $\sigma^2 < \infty$ and let $S_n = X_1 + X_2 + \cdots + X_n$. Then*

$$\frac{S_n - nm}{\sqrt{n\sigma^2}} \to N(0,1) \text{ as } n \to \infty.$$

The right side's first term of (5.3) converges with a normal distribution from this theorem. The right side's second term of (5.3) is χ^2 distribution and this term converges with a normal distribution, too. The right side converges with a normal distribution, because a normal distribution has an additivity property. So the distribution of the strategy converges with log-normal distribution. □

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